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Types of stationary points in a variational formulation of shallow-water flows

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Summary

Free-surface gravity flows are stationary points of a functional J when the problem is formulated variationally.

Here we are concerned with the problem of determining the nature of the stationary point, that is, whether it is a minimum, a maximum, a saddle point or whether a singularity occurs. This is a problem of both theoretical and computational importance.

Within a variational approximation of shallow-water type developed by the authors, we prove some new results on the problem. The analysis is carried out by studying the second variation of the functional J and the corresponding Jacobi's equation.

Reference is also made to numerical experiments which confirm the findings. The experiments also suggest that such findings may well extend to flows outside the class of shallow-water flows governed by the model used in the analysis.

1. Introduction

Variational formulations of free-surface flows have become increasingly popular in recent years. In such formulations, solutions of the physical problem are stationary points of a governing functional J. One of the principal reasons for this development is that variational principles may effectively be used to treat the non-linearities involved when computing the stationary point numerically. This is normally achieved in conjunction with numerical techniques such as the Finite Element Method, the Kantorovich Method, etc.

Steady ideal flows with a free surface under gravity were formulated variationally by O'Carroll and Harrison [1] in 1976. They expressed the problem in terms of a functional $J(h(x), \psi(x, y))$, where $h(x)$ determines the free-surface position and $\psi(x, y)$ is a volumetric stream function that governs the internal flow problem. For the last few years these variational formulations have become an established approach to solving free-surface gravity flows numerically (e.g. [2,3,4,10], [11]). Iterative techniques are invariably used and because of their obvious advantages variants of Newton's Method are widely utilised.

The problem of determining a priori whether the stationary point is a minimum, a maximum or of mixed type (saddle) requires attention. This is a topic of obvious

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theoretical importance in its own right. Moreover, from the computational point of view, information about the nature of the stationary point to be computed numerically may be relevant. For instance, if the stationary point is known to be a minimum, the computational problem is that of optimisation and any library subroutine may prove satisfactory. For saddle points however, the situation is somewhat different and not all optimisation techniques can be used.

Analysis of the second variation of J, denoted by $\delta^2 J$, can provide the required information regarding the nature of the stationary point of J, except for the case when $\delta^2 J$ is identically zero. But even the singular case gives valuable information, since this corresponds to a singularity of the Hessian matrix in the discrete version of the problem. It is then expected that any Newton's method will experience ill-conditioning when attempting to compute the stationary point.

For fixed-boundary problems it is well known that the functional takes a proper minimum at the solution ψ. However, the variation of the boundary position is more complex. Among uniform flows this may be minimum-like for shallow (rapid) flows but maximum-like for deep (tranquil) flows [12]. Some more general observations regarding the nature of the stationary point were made in [7]. These are based on numerical computation of the stationary point and subsequent evaluation of the Hessian matrix at the point. By testing the definiteness of this matrix at the computed solution we were able to find out its type. The limitations of this approach are obvious and an analysis of the continuous case in terms of the second variation is needed.

In this paper, which is based on the thesis [3] (see Chapter 6), we present some new results of the continuous problem. The analysis is carried out by studying the quadratic functional $\delta^2 J$. The problem is expressed in terms of an approximation of shallow-water type [5] which facilitates the task of deriving an explicit expression for the second variation of J. The conclusions are thus strictly applicable to shallow-water flows only. Numerical evidence however, suggests that some of the results may extend to more general flows.

It is shown that no stationary point can be a maximum and therefore the only possible stationary points are minima and of mixed type. A reference depth h_0 and a reference wavelength λ_0 are introduced. It is found that every flow whose free-surface height does not exceed h_0 is a minimum. The length λ_0 determines locations of the end boundary for which a wave of infinitesimal amplitude and wavelength λ_0 makes $\delta^2 J$ singular. This is significant for numerical computations. We also show that waves whose surface profiles lie entirely above h_0 are (i) saddle points in a channel of length greater than L_M and (ii) minima in a channel of length less than L_m. The lengths L_M and L_m are related to the amplitude of the wave and satisfy $L_m \leqslant \lambda_0/2 \leqslant L_M$.

There are still cases in the class of shallow-water flows that remain to be classified. It is also desirable to extend the analysis of the second variation of J for fuller two-dimensional free-surface flows.

2. Variational formulation of the problem

Here we are concerned with non-viscous flows with a free surface under gravity which are steady, two-dimensional, incompressible and irrotational. In terms of a volumetric stream function $\psi(x, y)$ both the bed and the free surface are streamlines. A typical flow domain is shown in Fig. 1, where H_0 measures the stagnation level (total head), l gives the length

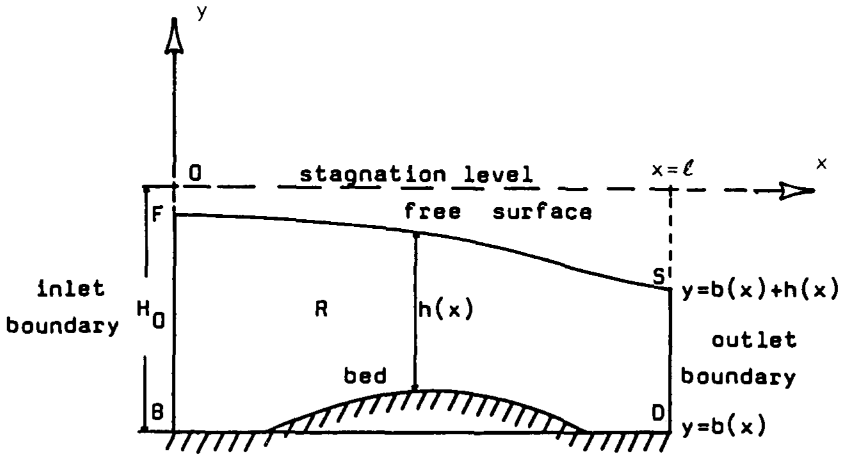

Figure 1. Flow domain with a free surface under gravity.

of the flow region in the x-direction, $b(x)$ is the position of the prescribed bed profile and $h(x)$ determines the position of the free surface.

Under the stated physical assumptions it can be shown [1] that the coupled problems of finding $h(x)$ and $\psi(x, y)$ are governed by the stationary conditions of the functional

$$J(h(x), \psi(x, y)) = \int_0^l \mathrm{d}x \int_{b(x)}^{b(x)+h(x)} \left\{ \tfrac{1}{2}(\nabla\psi)^2 - y \right\} \mathrm{d}y \tag{1}$$

with respect to both arguments, together with the constraints

$$\begin{aligned} &\psi = Q \quad \text{(the prescribed discharge) on BD,} \\ &\psi = 0 \quad \text{on} \quad \text{FS.} \end{aligned} \tag{2}$$

In expressions (1)–(2) all quantities have been non-dimensionalised with respect to length H_0 and time $(H_0/g)^{1/2}$, where g denotes the acceleration due to gravity. This variational formulation has been the starting point for several numerical computations (e.g. [2,3]).

Here we use an approximation of shallow-water type presented in [5], whereby $\psi(x, y)$ in Eqn. (1) is assumed to vary linearly in the y-direction from bed to surface. That is, ψ is given as follows

$$\psi(x, y) = Q\{b(x) + h(x) - y\}/h(x). \tag{3}$$

Direct evaluation of the integral with respect to y in (1) in terms of ψ as given by Eqn. (3) gives

$$J(h(x)) = \int_0^l F(x', h(x), h'(x)) \mathrm{d}x. \tag{4}$$

The function F in (4) is given by

$$F(x, h, h') = Q^2\{h'^2 + 3b'h' + 3b'^2 + 3\}/6h^2 - (h^2 + 2bh)/2 \tag{5}$$

where the argument x has been dropped, which will be done wherever convenient.

It is easy to show that the stationary equation of J as given by (4) and (5) is

$$2hh'' = h'^2 - 3\{b'^2 + b''h + 1 + 2h^2(b+h)/Q^2\}. \tag{6}$$

Equation (6) is the Euler-Lagrange equation which applies whatever end boundary conditions are used. If the end values of h are free, then there are additional natural boundary conditions as usual.

In the case of a flat bed solutions of this equation can be given in terms of cnoidal functions, but our main concern here is the characterisation of solutions regarded as stationary points of J. We shall carry out the analysis in terms of the second variation of J. This is a quadratic functional and will be derived in the following section.

3. The second variation of J for the shallow-water model

Following Gelfand and Fomin [8], the second variation of J as given by (4) can be expressed as

$$\delta^2 J(w) = \int_0^l (pw'^2 + qw^2)\mathrm{d}x$$

when $h(x)$ is given a variation $w(x)$ and the curves $h(x)$ have fixed end points, i.e. $h(0) = A$ and $h(l) = B$. The functions $w(x)$ satisfy the homogeneous boundary conditions. The functions $p(x)$ and $q(x)$ are as follows

$$p(x) = \tfrac{1}{2}F_{h'h'}, \quad q(x) = \tfrac{1}{2}\left(F_{hh} - \frac{\mathrm{d}}{\mathrm{d}x}F_{hh'}\right)$$

where $F = F(x, h(x), h'(x))$ is given by Eqn. (5). (There is a slight misprint in [8] in the expression for $q(x)$, page 102, Eqn. (11)).

The functions p and q are functions of x via h and as indicated previously, the argument x will be omitted wherever convenient. After differentiating we find that

$$p(x) = Q^2/6h,$$

$$q(x) = Q^2(2hh'' - 2h'^2 + 6 + 6b'^2 + 3b''h)/12h^3 - \tfrac{1}{2}. \tag{8}$$

Now we are in a position to prove the first result.

Theorem 1: *Within the stated shallow-water model with positive flow depth $h(x)$ for a channel with arbitrary bed profile and fixed end depths $h(0)$, $h(l)$ a stationary point of J can not be a maximum.*

PROOF: The proof follows immediately from the theory of second variations ([8] page 115), since a necessary condition for a solution $h(x)$ to give a maximum for J is that $p(x) \leqslant 0$, the Legendre condition. In fact, the strengthened Legendre's condition for a minimum ($p(x) > 0$) is satisfied for every solution $h(x)$, but this is neither necessary nor sufficient on its own for a minimum.

Before proceeding we recall some standard definitions and theorems about the theory of the second variation.

DEFINITION 1: The Jacobi's equation of J is defined as

$$-\frac{\mathrm{d}}{\mathrm{d}x}(pw') + qw = 0.$$

In fact, this is the Euler equation of the quadratic functional $\delta^2 J$, the second variation of J.

DEFINITION 2: A point $l_0 \in (0, l]$ is said to be conjugate to the point 0 (zero) if the initial-value problem

$$\begin{aligned} &-\frac{\mathrm{d}}{\mathrm{d}x}(pu') + qu = 0, \\ &u(0) = 0, \quad u'(0) = 1 \end{aligned} \tag{9}$$

has a solution with

$$u(l_0) = u(0) = 0.$$

The boundary conditions in (9) are different from the homogeneous boundary conditions to be satisfied by the admissible functions in the domain of $\delta^2 J$. This is to exclude trivial solutions of the Jacobi's equation.

The following are necessary conditions for a weak minimum for J.

(N1) Legendre's condition:
$p(x) \geqslant 0$ at every point of the curve $h(x)$.

(N2) Jacobi's condition:
The open interval $(0, l)$ has no points conjugate to zero.

The following is a sufficient set of conditions for a weak minimum.

(S1) Strengthened Legendre condition:
$p(x) > 0$ at every point of the curve $h(x)$.

(S2) The strengthened Jacobi's condition:
The interval $(0, l]$ contains no points conjugate to zero.

Both (S1) and (S2) have to be satisfied simultaneously. The proof is given in [8], pp. 116–117.

We now return to the specific problem of open-channel flow. It is worth remarking that

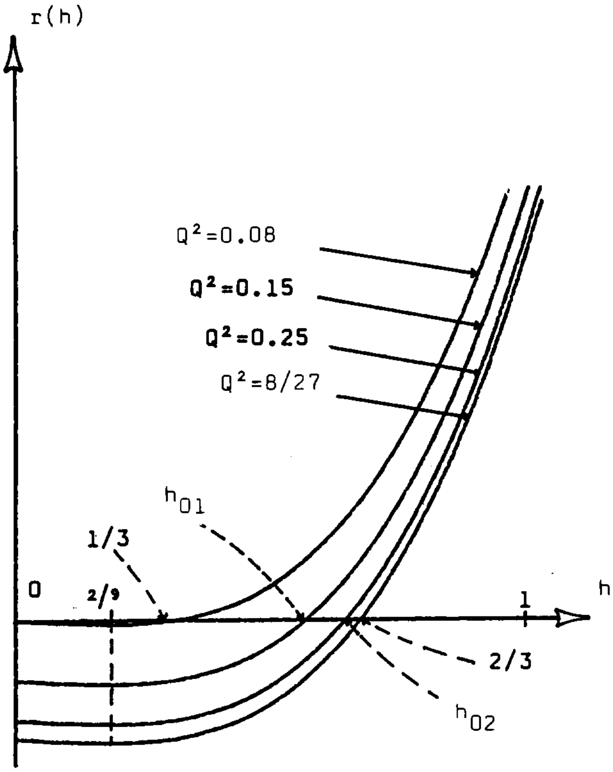

Figure 2. Plot of $r(h)$ as given by Eqn. (11).

since condition (S1) is satisfied in every case the nature of the extremal depends entirely on the zeros of the solution of initial-value problem (9).

Henceforth consider only the case of a flat bed with $b(x) = -1$ in the non-dimensionalised variables. Then by substituting $U(x) = u(x)/\sqrt{p(x)}$ the initial-value problem (9) is equivalent to

$$\begin{aligned} &U''(x) + k(h(x))U(x) = 0, \\ &U(0) = 0, \quad U'(0) = c_1 > 0 \end{aligned} \tag{10}$$

where

$$\begin{aligned} &k(h) = r(h)/s(h) \quad \text{with} \\ &r(h) = 6h^3 - 2h^2 - 3Q^2, \\ &s(h) = 4h^2Q^2/3. \end{aligned} \tag{11}$$

Notice that the sign and zeros of $k(h)$ are those of the cubic $r(h)$ (see Fig. 2) and the following properties are easily derived:

(i) $k(h)$ has only one real root h_0 for $0 < Q^2 < 8/27$ with

$$0 < S_1 < Q^{2/3} < h_0 < 2/3 < S_2 < 1, \tag{12}$$

where S_1 and S_2 are the two real solutions in (0, 1) of the cubic $2h^2(h-1)+Q^2=0$ governing uniform horizontal flows; S_1 is the depth of rapid uniform flow and S_2 the depth of the tranquil uniform flow. For $Q^2=8/27$, $S_1=Q^{2/3}=h_0=S_2=2/3$, and hence h_0 lies in $(1/3, 2/3]$.

(ii) Given two values of the discharge Q_1 and Q_2 satisfying $0<Q_1^2<Q_2^2<8/27$ we have that $1/3<h_{01}<h_{02}<2/3$ where h_{01} and h_{02} are the real roots of $k(h)$ for Q_1^2 and Q_2^2 respectively.

(iii) $k(h)<0$ for $0<h<h_0$ and $k(h)>0$ for $h_0<h<1$. In particular $0<k(h_1)<k(h_2)$ for $h_0<h_1<h_2$.

It is fortunate that the use of the stationary equation (6) leads to an expression for $k(h)$ that is independent of derivative terms. This facilitates the analysis of the variable-coefficient initial-value problem (10) considerably. For the raised-bed case however, the situation is different and the analysis more complicated.

Definition 2, concerning the existence (or non-existence) of conjugate points to zero, will become of crucial importance in determining the nature of the stationary point. The problem will be reduced to a problem of zeros of solutions of the initial-value problem (10).

4. Classification of stationary points

From the result of Theorem 1 we know that a non-singular stationary point is either a minimum or a saddle. The method of classifying relies upon the features of the variable coefficient $k(h(x))$ in the initial-value problem (10). As observed the main characteristics such as sign and zeros depend on the cubic $r(h)$ (see Fig. 2).

There are two principal cases given by intervals $(0, h_0)$ where k is negative and $(h_0, 1)$ where k is positive. The point h_0, where k vanishes, represents a reference depth of flow that separates types of stationary points in the present work. When k is non-positive we are able to classify all flows lying entirely below h_0 as minima. This result is independent of the channel length l. When k is positive we define a length function $L(h)=\pi/\sqrt{k(h)}$ in the interval $(h_0, 1)$, i.e.

$$L(h)=2\pi hQ/\left[3(6h^3-2h^2-3Q^2)\right]^{1/2}$$
$$\text{for } h_0<h_m<h(x)<h_M<1. \tag{13}$$

Here h_m and h_M denote the minimum and maximum depth of flow for $h(x)$. Figure 3 illustrates the function $L(h)$.

A solution $h(x)$ satisfying the restriction imposed in (13) will have two reference lengths $L_m=L(h_M)$ and $L_M=L(h_m)$ associated with it (note the reversed subscipts). h_m represents the trough and h_M the crest and then the classification will depend upon the channel length l and the bounds L_m and L_M.

A special case is found when the wave amplitude tends to zero. In this case h_m and h_M tend to the uniform flow solution S_2 from below and above respectively. By using the cubic $2h^3-2h^2+Q^2=0$, which is satisfied by S_2 we have

$$L(S_2)=\tfrac{1}{2}\lambda_0=\frac{\pi Q}{3\left(S_2-\frac{2}{3}\right)^{1/2}} \tag{14}$$

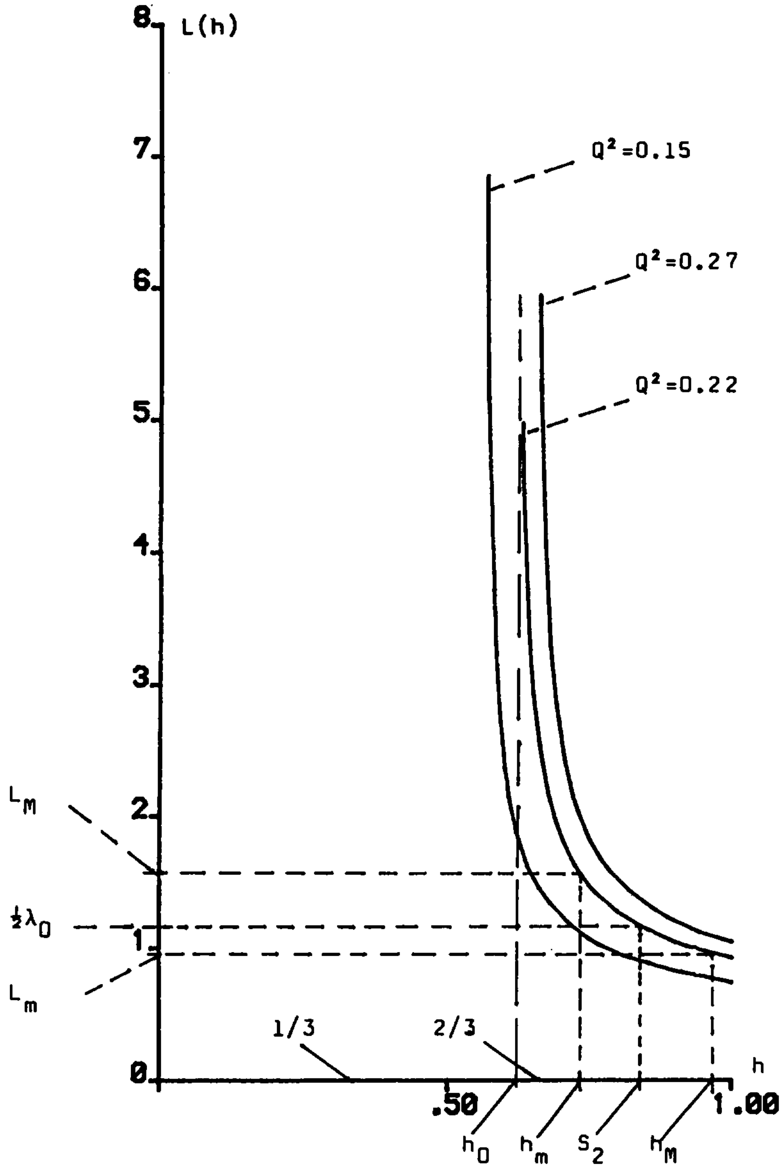

Figure 3. Plot of function $L(h)$ as given by Eqn. (13).

where λ_0 is the length of a wave of infinitesimal amplitude derived in [5] for this shallow-water model. Clearly $L_m = L_M = \frac{1}{2}\lambda_0$ in this case. In general $L_m < \frac{1}{2}\lambda_0 < L_M$ as illustrated in Fig. 3.

THEOREM 2: *Classification of stationary points of J for the shallow-water model with flat bed and fixed end depths* $h(0)$, $h(a)$:

(*i*) *Any solution* $h(x)$ *satisfying* $h(x) \leqslant h_0$ *throughout* $[0, l]$ *gives a minimum for J.*

(*ii*) *Any solution* $h(x)$ *satisfying* $h_0 < h_m \leqslant h(x) \leqslant h_M < 1$ *for all x in* $[0, l]$ *gives a saddle point for J on a channel of length* $l > L_M$.

(*iii*) *Any solution* $h(x)$ *as defined in* (*ii*) *gives a minimum for* J *on a channel of length* $l < L_m$.

(*iv*) *The stationary point* $h(x)$ *as defined in* (*ii*) *is singular for some* l *in* $[L_m, L_M]$.

In the above, a minimum means a weak or strong local minimum.

PROOF:

(i) Since condition (S1) is satisfied trivially only the strengthened Jacobi's condition has to be established. Thus we only need show that the solution of I.V.P. (10) does not vanish in $(0, l]$.

From hypothesis $h(x) \leqslant h_0$, for all x in $[0, l]$ and therefore $k(x) \leqslant 0$ (property iii) throughout the interval $[0, l]$. From the theory of differential equations, no nontrivial solution of the equation $U''(x) + k(x)U = 0$ has more than one zero in $[0, l]$ and therefore the solution to I.V.P. (10) does not vanish in $(0, L]$ (see [9]).

(ii) To prove that the solution $h(x)$ as in the hypothesis of the theorem is a saddle point for J when l is greater than L_M we only need show that $h(x)$ is not a minimum (Theorem 1). To this end it is enough to show that the Jacobi's necessary condition (N2) is not satisfied i.e. that the solution to I.V.P. (10) has at least one zero in the open interval $(0, l)$.

Since for all x in $[0, l]$ $h(x) \geqslant h_m$ from property iii, it follows that $k(x) \geqslant k(h_m)$ throughout $[0, l]$. Sturm's Comparison Theorems then show that any solution of the differential equation

$$U'' + k(x)U = 0, \quad x \in [0, l]$$

must vanish between two successive zeros of the solution of the differential equation

$$U'' + k(h_m)U = 0, \quad x \in [0, l]$$

and therefore vanishes in any interval $[0, l]$ with $l > \pi/\sqrt{k(h_m)} = L_M$.

(iii) From the hypothesis of the theorem $h(x) \leqslant h_M$ for all x in $[0, l]$. Thus $k(h(x)) \leqslant k(h_M)$, from property iii.

Then the solution to I.V.P. (10) does not vanish anywhere in $(0, L]$ when $l < \pi/\sqrt{k(h_M)} = L_m$, by Sturm's Comparison Theorem.

Therefore the strengthened Jacobi's sufficient condition is satisfied and the result follows.

(iv) The first zero of $U(x)$ for positive x from (10) occurs at $x = l$ for some l in $[L_m, L_M]$, then the second variation of J becomes zero [8] and therefore the solution $h(x)$ as defined in the hypothesis of the theorem is a singular point.

COROLLARY 1: *Uniform flow solutions for* $0 < Q^2 < 8/27$.

The uniform rapid solution S_1 is a minimum for J for all channel lengths. The tranquil uniform solution S_2 is a saddle, singular or minimum point for J according as $l > \frac{1}{2}\lambda_0$, $l = \frac{1}{2}\lambda_0$, or $l < \frac{1}{2}\lambda_0$ respectively.

The result follows immediately from Theorem 2 and observations on the function $L(h)$ illustrated in Fig. 3.

COROLLARY 2: *The critical uniform solution.*

The critical uniform solution $h(x) = \frac{2}{3}$ obtained when $Q^2 = 8/27$ is a minimum for J

for all channel lengths l. Note that this is for variations with fixed end points; variations of uniform-flow type would give a singular point for J [12].

In this case $h(x) = S_1 = S_2 = h_0 = \frac{2}{3}$ and therefore $k(h(x)) \equiv 0$. Then the result follows from Theorem 2, part (i).

5. Related numerical computations

Some numerical computations were carried out in order to verify in practice some of the theoretical predictions. As pointed out before, the theoretical results on the second variation for the single-layer theory apply directly to the general case only in the region of long waves, given the shallow-water type of approximation assumed in the model. In this context we computed a large number of cases related to some particular theoretical predictions. The algorithm NODE was used with one layer only [6].

The solution to be analysed is the tranquil uniform flow of depth S_2. According to Corollary 1 the nature of the solution is completely determined by the choice of channel length l in the computations. There are three cases, namely

(i) $l > \frac{1}{2}\lambda_0 = L(S_2)$. S_2 gives a saddle point,
(ii) $l = \frac{1}{2}\lambda_0$, S_2 is a singular point and
(iii) $l < \frac{1}{2}\lambda_0$, S_2 gives a minimum.

In all three cases we took a range of twenty values for the asymptotic level S_2 from $S_2 = 0.8566666$ (short-wave region) to $S_2 = 0.6766666$ (shallow water). The initial profile in the computations was taken precisely as the horizontal flow of depth S_2. This is particularly important for case (ii) where failure to compute the tranquil solution of known depth S_2 can not be attributed to the initial profile.

The computations for all twenty prescribed values of the depth S_2 in case (ii) failed to give the uniform flow of depth S_2. Other profiles were instead computed which exhibited small displacements at inlet and outlet. Also, computing time for this case (ii) was unusually large in comparison with the other two cases.

For the computations of case (i) we took the channel length L to be 20% greater than that of the singular case (ii), i.e. $l = 0.6\lambda_0$. The computed results show that for all the twenty prescribed values for S_2, the uniform tranquil solution of depth $h = S_2$ was accurately computed. Moreover, computing time was only about 5% of that for example (ii).

For case (iii) we took the channel length l to be 20% smaller than $L(S_2) = \frac{1}{2}\lambda_0$, i.e. $l = 0.4\lambda_0$. The uniform tranquil flow of depth $h = S_2$ was accurately computed in all cases and CPU time was slightly smaller than that of case (i).

Extensive numerical computations using the multilayer version of the present approximation [6] and a finite-element algorithm [3] confirm the previous observations. Moreover, the theoretical predictions of the shallow-water model seem to extend to deep-water flows. In this case the numerical ill-conditioning spreads to a neighbourhood of $\frac{1}{2}\lambda_0$. This is significant when computing non-linear water waves numerically, since they may be approached through a set of solutions with wavelengths developing from λ_0.

6. Conclusions

We have concluded that, within the approximate shallow-water theory, only minima and saddle points can occur, with or without singularity. This is true for any bed configuration

(Theorem 1). Numerical experiments suggest that this may be the case for other flows not governed by the shallow-water model. A reference depth h_0 satisfying $Q < S_1 < Q^{2/3} < h_0 < 2/3 < S_2 < 1$ has been found so that every flow $h(x)$ with $h(x) < h_0$ gives a minimum (Theorem 2, *i*). This includes all possible rapid uniform flows S_1 up to the maximum and flows converging to S_1 as $x \to \infty$. In addition, a range of wavy flows about S_2 are classified as saddle points for channel lengths l greater than a reference length L_M and as minima for l less than another length L_m. The lengths L_m and L_M are associated with the amplitude of each particular wave strictly above h_0.

A reference wavelength λ_0 has also been found in the process of analysis, which corresponds to the case of infinitesimal amplitude. The three lengths L_m, L_M and λ_0 satisfy the relation $L_m \leqslant \frac{1}{2}\lambda_0 \leqslant L_M < \infty$. The classification of all flows above h_0 has been linked to the channel length l. The results of Theorem 2 show that for l in the interval $(0, L_m)$ the stationary point is a minimum; for l in (L_m, ∞) the solution is a saddle point. However, for l in the interval $[L_m, L_M]$ we still do not know the nature of the stationary point in general. However, the case of the uniform tranquil flow is a singular point for $l = \frac{1}{2}\lambda_0$.

Numerical computations confirm the theoretical results and show that similar behaviour occurs outside the range of the shallow-water model. In particular computations for uniform flows fail if the computational channel length is chosen to coincide with a limiting wavelength, because the stationary point then becomes singular. Very-small-amplitude waves have been successfully computed by the authors but the convergence becomes more difficult as they approach the uniform flow.

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